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Some New Hybrid Block Methods for Solving Non-stiff Initial Value Problems of Ordinary Differential Equations

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ABSTRACT

In this paper, we derive some new k-step hybrid block methods (HBM) for the solution of first-order non-stiff initial value problems (IVPs) of ordinary differential equations (ODEs). A continuous hybrid multistep method (CHM) with variable coefficients is first developed using interpolation and collocation of a polynomial approximate solution. Discrete hybrid methods are then obtained by evaluating the CHM at some points of interest which give a class of discrete multistep methods that are implemented as hybrid block methods for $k = 3,4$ yielding HBM3 and HBM4 respectively. Properties of the HBM3 and HBM4 are investigated and the results yield orders seven and nine respectively and their convergence is established. Numerical performance of the block methods on some non-stiff ODEs shows that the methods compare favourably with some existing methods.

Keywords: Hybrid block methods, Linear non-stiff ODEs, Multistep collocation, ODEs.

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INTRODUCTION

We consider a numerical method for solving the general first-order initial value problem (IVP) of ordinary differential equations (ODEs) of the form

$$y' = f(x, y), y(x_0) = y_0 \quad (1.1)$$

whose solution $y(x)$ is sought in the range $a \le x \le b$, where $x_0 = a$ and b are finite, and f is a continuous function satisfying some Lipschitz condition for the existence and uniqueness of the solution. Badmus, Yahaya & Pam (2015) stated that problems in the form (1.1) have wide applications in engineering, physical sciences, medicine, molecular dynamics, quantum chemistry astrophysics, electronics and semi-discretization of wave equations etc.

Linear multistep methods (LMMs) are widely used for solving IVPs of first order ODEs. They are also applicable to solving higher order ODEs. Generally, LMMs are not self-starting especially when the step number $k > 1$ hence, need starting values from single-step methods like Euler's method and Runge-Kutta's family of methods (Aboiyar, Luga & Iyorter, 2015). The continued search for methods that enhance increased accuracy and computational efficiency has somehow seen the increased application of block methods as a set of simultaneous numerical integrators, which mostly do not require starters, especially when the step number $k > 1$ (see Abdullahi, Chollom & Yahaya, 2014; Akinfenwa, Jator & Yao, 2013; Badmus & Mshelia, 2012; Badmus, Yahaya & Pam, 2015; Jator & King, 2018; Kamoh, Gyemang & Soomiyol, 2017; Kumleng, Adee & Skwame, 2013; Odekunle, Adesanya & Sunday, 2012; Yakusak & Adeniyi, 2015, Yakusak, Emmanuel & Ogunniran, 2015). Such increased accuracy may come as a result of increasing the step number of the method which invariably increases the degree of the approximating polynomial.

Therefore, in this research, following the two- and three-step block hybrid methods developed by Badmus & Mshelia (2012), we obtain some new more accurate hybrid block methods for step numbers three and four, i.e., $k = 3,4$.

2. MATERIALS AND METHODS

In the spirit of Akinfenwa *et al.*, (2013), Badmus & Mshelia (2012), Jator & King (2018) and Kumleng *et al.*, (2013), the multistep collocation method we deploy in this research for solving (1.1) employs a k-step multistep method, having t interpolation points and m collocation points, which generally gives a continuous interpolant of the form

$$y(x) = \sum_{j=0}^{t-1} \phi_j(x) y_{n+j} + h \sum_{j=0}^{m-1} \varphi_j(x) f(\bar{x}_j, y(\bar{x}_j)) \qquad (2.1)$$

with $y(x)$ satisfying

$$\left.\begin{aligned} y(x_{n+j}) &= y_{n+j}, j \in \{0,1,\dots,t-1\} \\ y(\bar{x}_j) &= f(\bar{x}_j, y(\bar{x}_j)), j = 0,1,\dots,m-1 \end{aligned}\right\} \qquad (2.1a)$$

where $\phi_j(x)$ and $\varphi_j(x)$ are assumed polynomials of the form

$$\left.\begin{aligned} \phi_j(x) &= \sum_{i=0}^{t+m-1} \phi_{j,i+1} x^i, j \in \{0,1,\dots,t-1\} \\ h\varphi_j(x) &= h \sum_{i=0}^{t+m-1} \varphi_{j,i+1} x^i, j = 0,1,\dots,m-1 \end{aligned}\right\} \qquad (2.1b)$$

In eq (2.1a), x_{n+j} are t $(t > 0)$ arbitrary chosen interpolation points from $[x_n, x_{n+k}]$ whereas $\bar{x}_j$ are m $(m > 0)$ collocation points, also from $[x_n, x_{n+k}]$ (see Adee, 2002)

Further adapting Adee (2002), the following conditions are imposed on $\phi_j(x)$ and $\varphi_j(x)$:

$$\left.\begin{aligned} \phi_j(x_{n+i}) &= \delta_{ij}, i,j = 0,1,\dots,t-1 \\ h\varphi_j(x_{n+i}) &= 0, i = 0,1,\dots,t-1, j = 0,1,\dots,m-1 \end{aligned}\right\} \qquad (2.1c)$$

and

$$\left.\begin{aligned} \phi_j'(\bar{x}_i) &= 0, i = 0,1,\dots,m-1, j = 0,1,\dots,t-1 \\ h\varphi_j'(\bar{x}_i) &= \delta_{ij}, i,j = 0,1,\dots,m-1 \end{aligned}\right\} \qquad (2.1d)$$

Simplifying eqs. (2.1c-2.1d) leads to the matrix equation of the form $DC = I$ where I is an identity matrix of the same dimension $(t + m) \times (t + m)$ with D and C such that

$$D=\begin{bmatrix} 1 & x_n & x_n^2 & \cdots & x_n^{t+k-1} \\ 1 & x_{n+1} & x_{n+1}^2 & \cdots & x_{n+1}^{t+k-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{n+t-1} & x_{n+t-1}^2 & \cdots & x_{n+t-1}^{t+k-1} \\ 0 & 1 & 2\bar{x}_0 & \cdots & (t+m-1)\bar{x}_0^{(t+m-2)} \\ 0 & 1 & 2\bar{x}_1 & \cdots & (t+m-1)\bar{x}_1^{(t+m-2)} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 1 & 2\bar{x}_{m-1} & \cdots & (t+m-1)\bar{x}_{m-1}^{(t+m-2)} \end{bmatrix} \quad C=\begin{bmatrix} \phi_{01} & \cdots & \phi_{t-1,1} & h\varphi_{01} & \cdots & h\phi_{m-1,1} \\ \phi_{02} & \cdots & \phi_{t-1,2} & h\varphi_{02} & \cdots & h\phi_{m-1,2} \\ \phi_{03} & \cdots & \phi_{t-1,3} & h\varphi_{03} & \cdots & h\phi_{m-1,3} \\ \phi_{04} & \cdots & \phi_{t-1,4} & h\varphi_{04} & \cdots & h\phi_{m-1,4} \\ \vdots & & & & & \vdots \\ \phi_{0,t+m} & \cdots & \phi_{t-1,t+m} & h\varphi_{0,t+m} & \cdots & h\phi_{m-1,t+m} \end{bmatrix} \tag{2.2}$$

Thus, we can express eq (2.1) explicitly in the form

$$y(x) = (y_n, y_{n+1}, \ldots, y_{n+t-1}, f_n, f_{n+1}, \ldots, f_{n+m-1})^T C^T (1, x, x^2, \ldots, x^{t+m-1})^T \tag{2.3}$$

where, in (2.3), superscript T denotes the transpose of a matrix.

2.1 Derivation of new two hybrid block methods (HBM3 and HBM4)

Two cases of some new hybrid block methods are as follow:

Case 1 ($k = 3$)**:** Set the interpolation points $t = 3$ and collocation points $m = 5$ at $x_{n+j}, j = \frac{1}{2}, 1, \frac{3}{2}$ and $\bar{x}_j, j = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}$ respectively so that (2.1) becomes the continuous hybrid method (CHM3) of the form

$$y(x) = \sum_{i=1}^{3} \phi_{\frac{i}{2}}(x)\, y_{n+\frac{i}{2}} + h\sum_{i=1}^{5} \varphi_{\frac{i}{2}}(x)\, f_{n+\frac{i}{2}} \tag{2.4}$$

In (2.2), D takes the form

$$D=\begin{bmatrix} 1 & x_{n+\frac{1}{2}} & x_{n+\frac{1}{2}}^2 & x_{n+\frac{1}{2}}^3 & x_{n+\frac{1}{2}}^4 & x_{n+\frac{1}{2}}^5 & x_{n+\frac{1}{2}}^6 & x_{n+\frac{1}{2}}^7 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 \\ 1 & x_{n+\frac{3}{2}} & x_{n+\frac{3}{2}}^2 & x_{n+\frac{3}{2}}^3 & x_{n+\frac{3}{2}}^4 & x_{n+\frac{3}{2}}^5 & x_{n+\frac{3}{2}}^6 & x_{n+\frac{3}{2}}^7 \\ 0 & 1 & 2x_{n+\frac{1}{2}} & 3x_{n+\frac{1}{2}}^2 & 4x_{n+\frac{1}{2}}^3 & 5x_{n+\frac{1}{2}}^4 & 6x_{n+\frac{1}{2}}^5 & 7x_{n+\frac{1}{2}}^6 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 & 6x_{n+1}^5 & 7x_{n+1}^6 \\ 0 & 1 & 2x_{n+\frac{3}{2}} & 3x_{n+\frac{3}{2}}^2 & 4x_{n+\frac{3}{2}}^3 & 5x_{n+\frac{3}{2}}^4 & 6x_{n+\frac{3}{2}}^5 & 7x_{n+\frac{3}{2}}^6 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & 5x_{n+2}^4 & 6x_{n+2}^5 & 7x_{n+2}^6 \\ 0 & 1 & 2x_{n+\frac{5}{2}} & 3x_{n+\frac{5}{2}}^2 & 4x_{n+\frac{5}{2}}^3 & 5x_{n+\frac{5}{2}}^4 & 6x_{n+\frac{5}{2}}^5 & 7x_{n+\frac{5}{2}}^6 \end{bmatrix}$$

Applying Maple 18 software after some algebraic manipulations with $\xi = \frac{x-x_n}{h}$, the CHM (2.4) reduces to

$$y(\xi)=\begin{bmatrix}\frac{132}{17}\xi^7-\frac{1346}{17}\xi^6+\frac{5646}{17}\xi^5-\frac{12585}{17}\xi^4\\+\frac{3769}{4}\xi^3-\frac{5433}{8}\xi^2+\frac{8595}{34}\xi-\frac{4941}{136}\end{bmatrix}y_{n+\frac{1}{2}}+\begin{bmatrix}\frac{192}{17}\xi^7-\frac{1760}{17}\xi^6+\frac{6432}{17}\xi^5-\frac{12000}{17}\xi^4\\+716\xi^3-390\xi^2+\frac{1800}{17}\xi-\frac{189}{17}\end{bmatrix}y_{n+1}+$$

$$\begin{bmatrix}-\frac{324}{17}\xi^7+\frac{3106}{17}\xi^6-\frac{12078}{17}\xi^5+\frac{24585}{17}\xi^4\\-\frac{6633}{4}\xi^3+\frac{8553}{8}\xi^2-\frac{12195}{34}\xi+\frac{6589}{136}\end{bmatrix}y_{n+\frac{3}{2}}+\begin{bmatrix}\frac{281}{255}\xi^7-\frac{5809}{510}\xi^6+\frac{24811}{510}\xi^5-\frac{22669}{204}\xi^4\\+\frac{35113}{240}\xi^3-\frac{53129}{480}\xi^2+\frac{6029}{136}\xi-\frac{19551}{2720}\end{bmatrix}f_{n+\frac{1}{2}}+$$

$$\begin{bmatrix}\frac{2056}{255}\xi^7-\frac{20252}{255}\xi^6+\frac{81388}{255}\xi^5-\frac{34438}{51}\xi^4\\+\frac{24241}{30}\xi^3-\frac{32623}{60}\xi^2+\frac{3223}{17}\xi-\frac{8997}{340}\end{bmatrix}f_{n+1}+\begin{bmatrix}\frac{392}{85}\xi^7-\frac{3684}{85}\xi^6+\frac{14016}{85}\xi^5-\frac{5580}{17}\xi^4\\+\frac{3687}{10}\xi^3-\frac{4671}{20}\xi^2+\frac{1313}{17}\xi-\frac{3507}{340}\end{bmatrix}f_{n+\frac{3}{2}}+$$

$$\begin{bmatrix}-\frac{104}{255}\xi^7+\frac{908}{255}\xi^6-\frac{3212}{255}\xi^5+\frac{1198}{51}\xi^4\\-\frac{749}{30}\xi^3+\frac{907}{60}\xi^2-\frac{82}{17}\xi+\frac{213}{340}\end{bmatrix}f_{n+2}+\begin{bmatrix}\frac{11}{255}\xi^7-\frac{179}{510}\xi^6+\frac{601}{510}\xi^5-\frac{431}{204}\xi^4\\+\frac{523}{240}\xi^3-\frac{619}{480}\xi^2+\frac{55}{136}\xi-\frac{141}{2720}\end{bmatrix}f_{n+\frac{5}{2}} \quad (2.5)$$

Evaluating (2.5) at $\xi = 0,2,\frac{5}{2},3$ and its first derivative at $\xi = 0,3$ and after some algebraic manipulations, we obtain the formulae

$$y_{n+c_i} = y_n + h\sum_{i=0}^{6}\varphi_{\frac{i}{2}} f_{n+\frac{i}{2}} \quad (2.6)$$

Eq. (2.6) consists of six new hybrid methods whose coefficients are specified in Table 1.

Table 1: Coefficients of the hybrid method (2.6) for $k = 3$.

c_i	f_n $f_{n+\frac{5}{2}}$	$f_{n+\frac{1}{2}}$ f_{n+3}	f_{n+1} y_n	$f_{n+\frac{3}{2}}$ y_{n+c_i}	f_{n+2}
½	$\frac{19087}{120960}$ $\frac{263}{5040}$	$\frac{2713}{5040}$ $\frac{-863}{120960}$	$-\frac{15487}{40320}$ 1	$\frac{293}{945}$ 1	$\frac{-6737}{40320}$
1	$\frac{1139}{7560}$ $\frac{11}{315}$	$\frac{47}{63}$ $\frac{-37}{7560}$	$\frac{11}{2520}$ 1	$\frac{166}{945}$ 1	$\frac{-269}{2520}$
3⁄2	$\frac{137}{896}$ $\frac{27}{560}$	$\frac{81}{112}$ $\frac{-29}{4480}$	$\frac{1161}{4480}$ 1	$\frac{17}{35}$ 1	$\frac{-729}{4480}$
2	$\frac{143}{945}$ $\frac{8}{315}$	$\frac{232}{315}$ $\frac{-4}{945}$	$\frac{64}{315}$ 1	$\frac{752}{945}$ 1	$\frac{29}{315}$
5⁄2	$\frac{3715}{24192}$ $\frac{235}{1008}$	$\frac{725}{1008}$ $\frac{-275}{24192}$	$\frac{2125}{8064}$ 1	$\frac{125}{189}$ 1	$\frac{3875}{8064}$
3	$\frac{41}{280}$ $\frac{27}{35}$	$\frac{27}{35}$ $\frac{41}{280}$	$\frac{27}{280}$ 1	$\frac{34}{35}$ 1	$\frac{27}{280}$

Writing (2.6) using Table 1 in block form gives

$$A_0 Y_i = A_1 Y_{i-1} + h\left(B_0 F_i + B_1 F_{i-1}\right) \tag{2.7}$$

where i represents the block number, $Y_i = \left(y_{n+\frac{1}{2}}, y_{n+1}, y_{n+\frac{3}{2}}, y_{n+2}, y_{n+\frac{5}{2}}, y_{n+3}\right)^T$,

$Y_{i-1} = \left(y_{n-\frac{5}{2}}, y_{n-2}, y_{n-\frac{3}{2}}, y_{n-1}, y_{n-\frac{1}{2}}, y_n\right)^T$, $F_i = \left(f_{n+\frac{1}{2}}, f_{n+1}, f_{n+\frac{3}{2}}, f_{n+2}, f_{n+\frac{5}{2}}, f_{n+3}\right)^T$, $F_{i-1} =$ $\left(f_{n-\frac{5}{2}}, f_{n-2}, f_{n-\frac{3}{2}}, f_{n-1}, f_{n-\frac{1}{2}}, f_n\right)^T$, and T denotes transpose, A_0 is the identity matrix of size 6 while

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{19087}{120960} \\ 0 & 0 & 0 & 0 & 0 & \frac{1139}{7560} \\ 0 & 0 & 0 & 0 & 0 & \frac{137}{896} \\ 0 & 0 & 0 & 0 & 0 & \frac{143}{945} \\ 0 & 0 & 0 & 0 & 0 & \frac{3715}{24192} \\ 0 & 0 & 0 & 0 & 0 & \frac{41}{280} \end{bmatrix}, B_0 = \begin{bmatrix} \frac{2713}{5040} & -\frac{15487}{40320} & \frac{293}{945} & -\frac{6737}{40320} & \frac{263}{5040} & -\frac{863}{120960} \\ \frac{47}{63} & \frac{11}{2520} & \frac{166}{945} & -\frac{269}{2520} & \frac{11}{315} & -\frac{37}{7560} \\ \frac{81}{112} & \frac{1161}{4480} & \frac{17}{35} & -\frac{729}{4480} & \frac{27}{560} & -\frac{29}{4480} \\ \frac{232}{315} & \frac{64}{315} & \frac{752}{945} & \frac{29}{315} & \frac{8}{315} & -\frac{4}{945} \\ \frac{725}{1008} & \frac{2125}{8064} & \frac{125}{189} & \frac{3875}{8064} & \frac{235}{1008} & -\frac{275}{24192} \\ \frac{27}{35} & \frac{27}{280} & \frac{34}{35} & \frac{27}{280} & \frac{27}{35} & \frac{41}{280} \end{bmatrix}$$

Equation (2.7) is the new three-step hybrid block method (HBM3).

Case 2 ($k = 4$)**:** Here, we also set the interpolation points $t = 3$ exactly as in case 1 and collocation points $m = 7$ at $\bar{x}_j, j = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}$ respectively so that (2.1) becomes the continuous hybrid method (CHM4) of the form

$$y(x) = \sum_{i=1}^{3} \alpha_{\frac{i}{2}}(x)\, y_{n+\frac{i}{2}} + h\sum_{i=1}^{7} \beta_{\frac{i}{2}}(x)\, f_{n+\frac{i}{2}} \tag{2.8}$$

Here, the matrix D in (2.2) becomes

$$D = \begin{bmatrix} 1 & x_{n+\frac{1}{2}} & x^2_{n+\frac{1}{2}} & x^3_{n+\frac{1}{2}} & x^4_{n+\frac{1}{2}} & x^5_{n+\frac{1}{2}} & x^6_{n+\frac{1}{2}} & x^7_{n+\frac{1}{2}} & x^8_{n+\frac{1}{2}} & x^9_{n+\frac{1}{2}} \\ 1 & x_{n+1} & x^2_{n+1} & x^3_{n+1} & x^4_{n+1} & x^5_{n+1} & x^6_{n+1} & x^7_{n+1} & x^8_{n+1} & x^9_{n+1} \\ 1 & x_{n+\frac{3}{2}} & x^2_{n+\frac{3}{2}} & x^3_{n+\frac{3}{2}} & x^4_{n+\frac{3}{2}} & x^5_{n+\frac{3}{2}} & x^6_{n+\frac{3}{2}} & x^7_{n+\frac{3}{2}} & x^8_{n+\frac{3}{2}} & x^9_{n+\frac{3}{2}} \\ 0 & 1 & 2x_{n+\frac{1}{2}} & 3x^2_{n+\frac{1}{2}} & 4x^3_{n+\frac{1}{2}} & 5x^4_{n+\frac{1}{2}} & 6x^5_{n+\frac{1}{2}} & 7x^6_{n+\frac{1}{2}} & 8x^7_{n+\frac{1}{2}} & 9x^8_{n+\frac{1}{2}} \\ 0 & 1 & 2x_{n+1} & 3x^2_{n+1} & 4x^3_{n+1} & 5x^4_{n+1} & 6x^5_{n+1} & 7x^6_{n+1} & 8x^7_{n+1} & 9x^8_{n+1} \\ 0 & 1 & 2x_{n+\frac{3}{2}} & 3x^2_{n+\frac{3}{2}} & 4x^3_{n+\frac{3}{2}} & 5x^4_{n+\frac{3}{2}} & 6x^5_{n+\frac{3}{2}} & 7x^6_{n+\frac{3}{2}} & 8x^7_{n+\frac{3}{2}} & 9x^8_{n+\frac{3}{2}} \\ 0 & 1 & 2x_{n+2} & 3x^2_{n+2} & 4x^3_{n+2} & 5x^4_{n+2} & 6x^5_{n+2} & 7x^6_{n+2} & 8x^7_{n+2} & 9x^8_{n+2} \\ 0 & 1 & 2x_{n+\frac{5}{2}} & 3x^2_{n+\frac{5}{2}} & 4x^3_{n+\frac{5}{2}} & 5x^4_{n+\frac{5}{2}} & 6x^5_{n+\frac{5}{2}} & 7x^6_{n+\frac{5}{2}} & 8x^7_{n+\frac{5}{2}} & 9x^8_{n+\frac{5}{2}} \\ 0 & 1 & 2x_{n+3} & 3x^2_{n+3} & 4x^3_{n+3} & 5x^4_{n+3} & 6x^5_{n+3} & 7x^6_{n+3} & 8x^7_{n+3} & 9x^8_{n+3} \\ 0 & 1 & 2x_{n+\frac{7}{2}} & 3x^2_{n+\frac{7}{2}} & 4x^3_{n+\frac{7}{2}} & 5x^4_{n+\frac{7}{2}} & 6x^5_{n+\frac{7}{2}} & 7x^6_{n+\frac{7}{2}} & 8x^7_{n+\frac{7}{2}} & 9x^8_{n+\frac{7}{2}} \end{bmatrix}$$

Following the derivation in Case 1 yields the CHM4 of the form,

$$y(\xi)=\begin{bmatrix} \frac{56160}{58879}\xi^9-\frac{962112}{58879}\xi^8+\frac{7054032}{58879}\xi^7-\frac{28967008}{58879}\xi^6+\frac{73182606}{58879}\xi^5 \\ -\frac{117507516}{58879}\xi^4+\frac{119313022}{58879}\xi^3-\frac{73324116}{58879}\xi^2+\frac{24441480}{58879}\xi-\frac{3286548}{58879} \end{bmatrix} y_{n+\frac{1}{2}} +$$

$$\begin{bmatrix} \frac{163840}{58879}\xi^9-\frac{2678016}{58879}\xi^8+\frac{18518016}{58879}\xi^7-\frac{70680064}{58879}\xi^6+\frac{163000320}{58879}\xi^5 \\ -\frac{233808288}{58879}\xi^4+\frac{207099136}{58879}\xi^3-\frac{108692928}{58879}\xi^2+\frac{30723840}{58879}\xi-\frac{3586977}{58879} \end{bmatrix} y_{n+1} +$$

$$\begin{bmatrix} -\frac{220000}{58879}\xi^9+\frac{3640128}{58879}\xi^8-\frac{25572048}{58879}\xi^7+\frac{99647072}{58879}\xi^6-\frac{236182926}{58879}\xi^5 \\ +\frac{351315804}{58879}\xi^4-\frac{326412158}{58879}\xi^3+\frac{182017044}{58879}\xi^2-\frac{55165320}{58879}\xi+\frac{6932404}{58879} \end{bmatrix} y_{n+\frac{3}{2}} +$$

$$\begin{bmatrix} \frac{1378228}{11128131}\xi^9-\frac{880926}{412153}\xi^8+\frac{293390806}{18546885}\xi^7-\frac{19358559}{294395}\xi^6+\frac{1789208741}{10598220}\xi^5 \\ -\frac{130464265}{471032}\xi^4+\frac{64467190117}{222562620}\xi^3-\frac{1529938699}{8243060}\xi^2+\frac{3875338}{58879}\xi-\frac{161046283}{16486120} \end{bmatrix} f_{n+\frac{1}{2}} +$$

$$\begin{bmatrix} \frac{1594592}{1236459}\xi^9-\frac{133779664}{6182295}\xi^8+\frac{956019032}{6182295}\xi^7-\frac{543378172}{883185}\xi^6+\frac{1320509918}{883185}\xi^5 \\ -\frac{2022626341}{883185}\xi^4+\frac{27187290209}{12364590}\xi^3-\frac{10457166729}{8243060}\xi^2+\frac{23380173}{58879}\xi-\frac{423251181}{8243060} \end{bmatrix} f_{n+1} +$$

$$\begin{bmatrix} \frac{1341724}{1236459}\xi^9-\frac{21934946}{1236459}\xi^8+\frac{50649546}{412153}\xi^7-\frac{27746887}{58879}\xi^6+\frac{258437107}{235516}\xi^5 \\ -\frac{754642861}{471032}\xi^4+\frac{7229191835}{4945836}\xi^3-\frac{3964804813}{4945836}\xi^2+\frac{14102660}{58879}\xi-\frac{98081883}{3297224} \end{bmatrix} f_{n+\frac{3}{2}} +$$

$$\begin{bmatrix} -\frac{2297024}{11128131}\xi^9+\frac{4008832}{1236459}\xi^8-\frac{79663312}{3709377}\xi^7+\frac{13859296}{176637}\xi^6-\frac{92099876}{529911}\xi^5 \\ +\frac{42703672}{176637}\xi^4-\frac{2348055109}{11128131}\xi^3+\frac{45886906}{412153}\xi^2-\frac{1896365}{58879}\xi+\frac{1605908}{412153} \end{bmatrix} f_{n+2} +$$

$$\begin{bmatrix} \frac{81724}{1236459}\xi^9-\frac{1236814}{1236459}\xi^8+\frac{7888526}{1236459}\xi^7-\frac{3970051}{176637}\xi^6+\frac{34036625}{706548}\xi^5 \\ -\frac{92024513}{1413096}\xi^4+\frac{274445659}{4945836}\xi^3-\frac{47330877}{1648612}\xi^2+\frac{481374}{58879}\xi-\frac{3220893}{3297224} \end{bmatrix} f_{n+\frac{5}{2}} +$$

$$\begin{bmatrix} -\frac{18208}{1236459}\xi^9+\frac{265232}{1236459}\xi^8-\frac{2723384}{2060765}\xi^7+\frac{1330484}{294395}\xi^6-\frac{2782166}{294395}\xi^5 \\ +\frac{737219}{58879}\xi^4-\frac{129795839}{12364590}\xi^3+\frac{132587267}{24729180}\xi^2-\frac{88993}{58879}\xi+\frac{1476969}{8243060} \end{bmatrix} f_{n+3} +$$

$$\begin{bmatrix} \frac{17428}{11128131}\xi^9-\frac{135566}{6182295}\xi^8+\frac{2429254}{18546885}\xi^7-\frac{386123}{883185}\xi^6+\frac{9506429}{10598220}\xi^5 \\ -\frac{8272177}{7065480}\xi^4+\frac{215916373}{222562620}\xi^3-\frac{4047241}{8243060}\xi^2+\frac{8092}{58879}\xi-\frac{267133}{16486120} \end{bmatrix} f_{n+\frac{7}{2}} \tag{2.9}$$

Evaluating (2.9) at $\xi = 0,2,\frac{5}{2},3,\frac{7}{2},4$ and its first derivative at $\xi = 0,4$ and after some algebraic manipulations, we obtain the formulae

$$y_{n+c_i} = y_n + h\sum_{i=0}^{8}\varphi_{\frac{i}{2}} f_{n+\frac{i}{2}} \tag{2.10}$$

Eq. (2.10) consists of eight new hybrid methods whose coefficients are specified in Table 2.

Table 2: Coefficients of the hybrid method (2.10) for $k = 4$.

c_i	f_n / f_{n+3}	$f_{n+\frac{1}{2}}$ / $f_{n+\frac{7}{2}}$	f_{n+1} / f_{n+4}	$f_{n+\frac{3}{2}}$ / y_n	f_{n+2} / y_{n+c_i}	$f_{n+\frac{5}{2}}$
$\frac{1}{2}$	$\frac{1070017}{7257600}$	$\frac{2233547}{3628800}$	$\frac{2302297}{3628800}$	$\frac{2797679}{3628800}$	$\frac{-31457}{45360}$	$\frac{1573169}{3628800}$
	$\frac{-645607}{3628800}$	$\frac{156437}{3628800}$	$\frac{-33953}{7257600}$	1	1	
1	$\frac{32377}{226800}$	$\frac{22823}{28350}$	$\frac{-21247}{113400}$	$\frac{15011}{28350}$	$\frac{-2903}{5670}$	$\frac{9341}{28350}$
	$\frac{-15577}{113400}$	$\frac{953}{28350}$	$\frac{-119}{32400}$	1	1	
$\frac{3}{2}$	$\frac{12881}{89600}$	$\frac{35451}{44800}$	$\frac{1719}{44800}$	$\frac{39967}{44800}$	$\frac{-351}{560}$	$\frac{17217}{44800}$
	$\frac{-7031}{44800}$	$\frac{243}{6400}$	$\frac{-369}{89600}$	1	1	
2	$\frac{4063}{28350}$	$\frac{11288}{14175}$	$\frac{122}{14175}$	$\frac{16376}{14175}$	$\frac{-908}{2835}$	$\frac{4616}{14175}$
	$\frac{-1978}{14175}$	$\frac{488}{14175}$	$\frac{-107}{28350}$	1	1	
$\frac{5}{2}$	$\frac{41705}{290304}$	$\frac{115075}{145152}$	$\frac{3775}{145152}$	$\frac{159175}{145152}$	$\frac{-125}{9072}$	$\frac{85465}{145152}$
	$\frac{-24575}{145152}$	$\frac{5725}{145152}$	$\frac{-175}{41472}$	1	1	
3	$\frac{401}{2800}$	$\frac{279}{350}$	$\frac{9}{1400}$	$\frac{403}{350}$	$\frac{-9}{70}$	$\frac{333}{350}$
	$\frac{79}{1400}$	$\frac{9}{350}$	$\frac{-9}{2800}$	1	1	
$\frac{7}{2}$	$\frac{149527}{1036800}$	$\frac{408317}{518400}$	$\frac{24353}{518400}$	$\frac{542969}{518400}$	$\frac{343}{6480}$	$\frac{368039}{518400}$
	$\frac{261023}{518400}$	$\frac{111587}{518400}$	$\frac{-8183}{1036800}$	1	1	
4	$\frac{1978}{14175}$	$\frac{1978}{14175}$	$\frac{-1856}{14175}$	$\frac{20992}{14175}$	$\frac{-1816}{2835}$	$\frac{20992}{14175}$
	$\frac{-1856}{14175}$	$\frac{11776}{14175}$	$\frac{1978}{14175}$	1	1	

Writing (2.10) using Table 2 in block form gives

$$A_0 Y_i = A_1 Y_{i-1} + h\left(B_0 F_i + B_1 F_{i-1}\right) \tag{2.11}$$

where i represents the block number, $Y_i = \left(y_{n+\frac{1}{2}}, y_{n+1}, y_{n+\frac{3}{2}}, y_{n+2}, y_{n+\frac{5}{2}}, y_{n+3}, y_{n+\frac{7}{2}}, y_{n+4}\right)^T$, $Y_{i-1} =$ $\left(y_{n-\frac{7}{2}}, y_{n-3}, y_{n-\frac{5}{2}}, y_{n-2}, y_{n-\frac{3}{2}}, y_{n-1}, y_{n-\frac{1}{2}}, y_n\right)^T$,

$F_i = \left(f_{n+\frac{1}{2}}, f_{n+1}, f_{n+\frac{3}{2}}, f_{n+2}, f_{n+\frac{5}{2}}, f_{n+3}, f_{n+\frac{7}{2}}, f_{n+4}\right)^T$,

$F_{i-1} = \left(f_{n-\frac{7}{2}}, f_{n-3}, f_{n-\frac{5}{2}}, f_{n-2}, f_{n-\frac{3}{2}}, f_{n-1}, f_{n-\frac{1}{2}}, f_n\right)^T$, and T denotes transpose, A_0 is the identity matrix of size 8 while

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1070017}{7257600} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{32377}{226800} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{12881}{89600} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4063}{28350} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{41705}{290304} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{401}{2800} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{149527}{1036800} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1978}{14175} \end{bmatrix},$$

$$B_0 = \begin{bmatrix} \frac{2233547}{3628800} & -\frac{2302297}{3628800} & \frac{2797679}{3628800} & -\frac{31457}{45360} & \frac{1573169}{3628800} & -\frac{645607}{3628800} & \frac{156437}{3628800} & -\frac{33953}{7257600} \\ \frac{22823}{28350} & -\frac{21247}{113400} & \frac{15011}{28350} & -\frac{2903}{5670} & \frac{9341}{28350} & -\frac{15577}{113400} & \frac{953}{28350} & -\frac{119}{32400} \\ \frac{35451}{44800} & \frac{1719}{44800} & \frac{39967}{44800} & -\frac{351}{560} & \frac{17217}{44800} & -\frac{7031}{44800} & \frac{243}{6400} & -\frac{369}{89600} \\ \frac{11288}{14175} & \frac{122}{14175} & \frac{16376}{14175} & -\frac{908}{2835} & \frac{4616}{14175} & -\frac{1978}{14175} & \frac{488}{14175} & -\frac{107}{28350} \\ \frac{115075}{145152} & \frac{3775}{145152} & \frac{159175}{145152} & -\frac{125}{9072} & \frac{85465}{145152} & -\frac{24575}{145152} & \frac{5725}{145152} & -\frac{175}{41472} \\ \frac{279}{350} & \frac{9}{1400} & \frac{403}{350} & -\frac{9}{70} & \frac{333}{350} & \frac{79}{1400} & \frac{9}{350} & -\frac{9}{2800} \\ \frac{408317}{518400} & \frac{24353}{518400} & \frac{542969}{518400} & \frac{343}{6480} & \frac{368039}{518400} & \frac{261023}{518400} & \frac{111587}{518400} & -\frac{8183}{1036800} \\ \frac{11776}{14175} & -\frac{1856}{14175} & \frac{20992}{14175} & -\frac{1816}{2835} & \frac{20992}{14175} & -\frac{1856}{14175} & \frac{11776}{14175} & \frac{1978}{14175} \end{bmatrix}$$

Equation (2.11) is the new four-step hybrid block method (HBM4)

3. Analysis of the New Methods HBM3 and HBM4

In this section, the orders and error constants, consistency, zero-stability, linear stability and region of absolute stability of the block methods (2.7) and (2.11) is presented.

3.1 Order of Accuracy and Error Constants of HBM3 AND HBM4

Following Odekunle *et al.*, (2012), the methods HBM3 and HBM4 are of order $p = 7 \geq 1$ and $p = 9 \geq 1$ with error constants in Tables 3 and 4 respectively.

Table 3: Features of HBM3

Scheme	Order	Error Constant
$y_{n+\frac{1}{2}}$	7	$\frac{275}{6193152}$
y_{n+1}	7	$\frac{1}{30240}$
$y_{n+\frac{3}{2}}$	7	$\frac{9}{229376}$
y_{n+2}	7	$\frac{1}{30240}$
$y_{n+\frac{5}{2}}$	7	$\frac{275}{6193152}$
y_{n+3}	8	$\frac{28823}{120422400}$

Table 4: Features of the HBM4

Scheme	Order	Error Constant
$y_{n+\frac{1}{2}}$	9	$\frac{8183}{1061683200}$
y_{n+1}	9	$\frac{9}{1433600}$
$y_{n+\frac{3}{2}}$	9	$\frac{25}{3670016}$
y_{n+2}	9	$\frac{47}{7257600}$
$y_{n+\frac{5}{2}}$	9	$\frac{25}{3670016}$
y_{n+3}	9	$\frac{9}{1433600}$
$y_{n+\frac{7}{2}}$	9	$\frac{8183}{1061683200}$
y_{n+4}	10	$-\frac{37}{14968800}$

3.2 Consistency of the HBM3 AND HBM4

From Tables 3 and 4, the block methods HBM3 and HBM4 are consistent since they have orders $p = 7 > 1$and $p = 9 > 1$ respectively (Henrici, 1962).

3.3 Zero Stability of the HBM3 AND HBM4

The methods (2.7) and (2.11) are said to be zero stable if the roots $Z_s, s = 1,2, \dots . n$ of the first characteristics polynomial $\rho(\lambda)$ defined by $\rho(\lambda) = \det[\lambda A_1 - A_0] = 0$ satisfy $|Z_s| \le 1, s = 1,2, \dots, k$ where A_1 and A_0 were defined in (2.7) and (2.11).

Thus, $\rho(\lambda) = \lambda^6 - \lambda^5 = 0 \Rightarrow \lambda^5(\lambda - 1) = 0, \lambda = 1, \lambda = 0$ (5 times) and

$\rho(\lambda) = \lambda^8 - \lambda^7 = 0 \Rightarrow \lambda^7(\lambda - 1) = 0, \lambda = 1, \lambda = 0$ (7 times) respectively.

Accordingly, the HBM3 and HBM4 are zero stable, hence convergent (Fatunla, 1991; Henrici, 1962).

3.4 Linear Stability of the HBM3 and HBM4

Based on Akinfenwa *et al.*, (2013), the stability functions for each of the methods HBM3 and HBM4 is given by $\mu(z) = -(A_1 - zB_1)^{-1}(A_0 - zB_0)$ where in each case, A_1, A_0, B_1 and B_0 are defined in (2.7) and (2.11). These yields

$$\mu(z) = -\frac{15z^6 + 147z^5 + 812z^4 + 2940z^3 + 7000z^2 + 10080z + 6720}{15z^6 - 147z^5 + 812z^4 - 2940z^3 + 7000z^2 - 10080z + 6720} \tag{3.1}$$

and

$$\mu(z) = -\frac{\left(\begin{array}{l} 420z^8 + 4566z^7 + 29531z^6 + 134568z^5 + 448980z^4 \\ +1088640z^3 + 1834560z^2 + 1935360z + 967680 \end{array}\right)}{\left(\begin{array}{l} 420z^8 - 4566z^7 + 29531z^6 - 134568z^5 + 448980z^4 \\ -1088640z^3 + 1834560z^2 - 1935360z + 967680 \end{array}\right)} \tag{3.2}$$

for HBM3 and HBM4 respectively.

3.5 Regions of Absolute Stability of the HBM3 AND HBM4.

In a similar vein, using (3.1) and (3.2), the stability polynomials given by

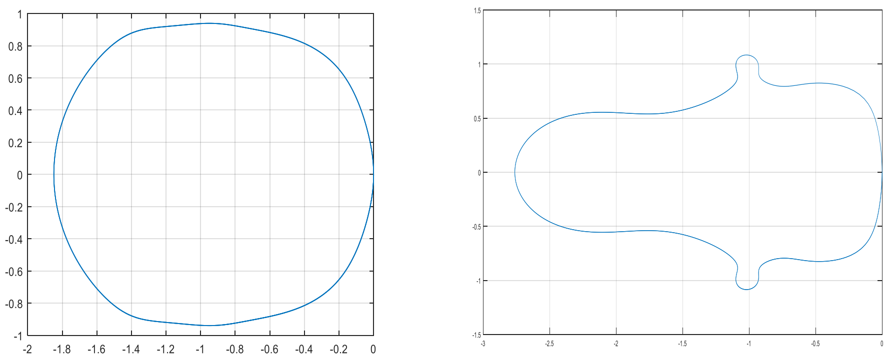
$$-z^5\left(\frac{431}{13440}w^6+\frac{157}{13440}w^5\right)-z^4\left(\frac{4463}{67200}w^5-\frac{4463}{67200}w^6\right)-z^3\left(\frac{18853}{33600}w^6+\frac{10547}{33600}w^5\right)$$
$$-z^2\left(\frac{1381}{1680}w^5-\frac{1381}{1680}w^6\right)-z\left(\frac{461}{280}w^6+\frac{379}{280}w^5\right)+w^6-w^5 \quad (3.3)$$

and

$$-z^7\left(\frac{281}{40320}w^8+\frac{199}{80640}w^7\right)-z^6\left(\frac{1566947}{101606400}w^7-\frac{1566947}{101606400}w^8\right)$$
$$-z^5\left(\frac{590221}{3175200}w^8+\frac{585763}{6350400}w^7\right)-z^4\left(\frac{3531473}{10886400}w^7-\frac{3531473}{10886400}w^8\right)$$
$$-z^3\left(\frac{38407}{28350}w^8+\frac{50761}{56700}w^7\right)-z^2\left(\frac{366679}{226800}w^7-\frac{366679}{226800}w^8\right)$$
$$-z\left(\frac{30328}{14175}w^8+\frac{26372}{14175}w^7\right)+w^8-w^7 \quad (3.4)$$

for HBM3 and HBM4 respectively.

To obtain the stability regions of the HBM3 (2.7) and HBM4 (2.11), the stability polynomials (3.3) and (3.4) and their first derivatives are used to plot the region of absolute stability as indicated in in Figure 1 (a) and (b) respectively.



(a) (b)

Fig. 1: Region of Absolute Stability (a) HBM3 and (b) HBM4.

Note that the regions indicate that the methods, i.e., HBM3 and HBM4 are absolutely unstable.

4. Numerical Experiments

The following numerical experiments are performed with the aid of MATLAB $R2015a$ software package to further affirm the earlier established convergence of the absolute errors of the new HBM3 and HBM4. To test the accuracy of the methods, we compare the absolute errors of the methods using the following notations in our tables below.

HBM3: new hybrid block method (Case 1)

HBM4: new hybrid block method (Case 2)

BHBM2: Two-step method by Badmus & Mshelia (2012)
BHBM3: Three-step method by Badmus & Mshelia (2012)
YKA: Yakusak & Adeniyi (2015)
YA: Yahaya (2004)
4S2HBM: Four-step two off -step Hybrid Block Method by Yakusak *et al.*, (2015)
ACY: Abdullahi, Chollom & Yahaya (2014)
OAS: Odekunle, Adesanya & Sunday (2012)
BYP: Badmus, Yahaya & Pam (2015)
KGS: Kamoh, Gyemang & Soomiyol (2017)

Note: in Tables 5-10, we have used the notation $a(-b) := a \times 10^{-b}$

Example 1

$xy' + 2y = 4x^2, y(1) = 2, h = 0.01$ (Exact solution: $y(x) = x^2 + \frac{1}{x^2}$).

Source: Badmus & Mshelia (2012). Results are shown in Table 5.

Table 5. Comparison of Absolute Errors for Example 1

x	HBM3	HBM4	$BHBM2$	$BHBM3$
1.1	6.2173(−15)	4.4409(−16)	3.8299(−07)	1.1799(−07)
1.2	9.2282(−13)	5.7732(−15)	3.3556(−08)	4.8444(−07)
1.3	1.4467(−11)	1.8696(−13)	5.3331(−08)	3.5903(−06)
1.4	9.3087(−11)	1.9593(−12)	6.5367(−08)	3.1036(−06)
1.5	3.7263(−10)	1.1354(−11)	1.4440(−09)	2.7314(−06)
1.6	1.1131(−09)	4.5551(−11)	3.3000(−08)	8.3100(−07)

Example 2

$y' = 5y, y(0) = 1, h = 0.01$ (Exact solution: $y(x) = e^{5x}$)

Source: Yakusak & Adeniyi (2015) and Yahaya (2004). Results are shown in Table 6.

Table 6: Comparison of Absolute Errors for Example 2

x	HBM3	HBM4	YKA	YA
0.01	1.6(−15)	1.1(−15)	0.0000	6.2(−10)
0.02	3.1(−15)	8.9(−16)	0.0000	1.1(−09)
0.03	4.7(−15)	2.2(−16)	0.0000	1.3(−09)
0.04	6.7(−15)	6.7(−16)	0.0000	1.6(−10)
0.05	8.7(−15)	1.3(−15)	0.0000	1.3(−09)
0.06	1.1(−14)	4.4(−16)	1.0(−9)	4.2(−10)
0.07	1.4(−14)	2.2(−16)	1.0(−9)	2.4(−10)
0.08	1.6(−14)	4.4(−16)	1.0(−9)	2.4(−09)
0.09	1.9(−14)	1.1(−15)	0.00000	2.5(−09)

Example 3

$y' = -y, y(0) = 1, 0 \le x \le 1, h = 0.1$ (Exact solution: $y(x) = e^{-x}$)

Source: Yakusak *et al.*,(2015); Abdullahi, Chollom & Yahaya (2014) (see Table 7).

Table 7: Comparison of Absolute Errors for Example 3

x	HBM3	HBM4	4S2HBM	ACY
0.1	2.6(−13)	4.4(−16)	0.00000	0.00000
0.2	4.7(−13)	7.8(−16)	0.00000	0.00000
0.3	6.3(−13)	1.1(−15)	0.00000	2.0(−10)
0.4	7.7(−13)	1.3(−15)	1.0(−09)	1.0(−10)
0.5	8.7(−13)	1.4(−15)	1.0(−09)	6.0(−10)
0.6	9.4(−13)	1.7(−15)	1.2(−03)	7.0(−10)
0.7	9.9(−13)	1.7(−15)	1.0(−09)	4.0(−10)
0.8	1.0(−12)	1.7(−15)	1.0(−09)	8.0(−10)
0.9	1.0(−12)	1.8(−15)	7.1(−09)	9.0(−10)
1.0	1.1(−12)	1.8(−15)	1.0(−09)	8.0(−10)

Example 4

Consider a linear first-order IVP,

$y' = xy, y(0) = 1, h = 0.1, 0 \le x \le 2$ (Exact solution: $y(x) = e^{\left(\frac{x^2}{2}\right)}$).

Source: Odekunle, *et al.*, (2012) and Badmus *et al.*, (2015). Results are shown in Table 8.

Table 8: Comparison of Absolute Errors for Example 4

x	HBM3	HBM4	OAS	BYP
0.1	3.7954(−11)	7.1565(−13)	5.2398(−07)	1.2313(−09)
0.2	8.3136(−11)	1.6251(−12)	1.6913(−07)	1.2343(−09)
0.3	1.4096(−10)	2.8595(−12)	8.7243(−07)	1.2814(−09)
0.4	2.1862(−10)	4.6037(−12)	3.0098(−06)	9.0100(−12)
0.5	3.2622(−10)	7.1283(−12)	1.7466(−06)	3.3949(−09)
0.6	4.7818(−10)	1.0837(−11)	4.1710(−06)	3.5428(−09)
0.7	6.9563(−10)	1.6340(−11)	9.6465(−06)	3.8244(−09)
0.8	1.0097(−09)	2.4567(−11)	6.7989(−06)	3.8275(−10)
0.9	1.4667(−09)	3.6946(−11)	1.2913(−05)	1.3275(−08)
1.0	2.1361(−09)	5.5682(−11)	2.6575(−05)	1.4427(−08)

Example 5

Consider a linear first-order IVP,

$$y' = x - y, y(0) = 0, 0 \le x \le 1, h = 0.1$$ (Exact solution: $y(x) = x + e^{-x} - 1$).

Source: Kamoh *et al.,* (2017). Results are shown in Table 9.

Table 9: Comparison of Absolute Errors for Example 5

x	HBM3	HBM4	KGS
0.1	2.58(−13)	5.02(−16)	1.05(−09)
0.2	4.68(−13)	8.81(−16)	6.20(−10)
0.3	6.35(−13)	1.12(−15)	7.96(−10)
0.4	7.66(−13)	1.55(−15)	3.86(−10)
0.5	8.66(−13)	1.58(−15)	1.42(−09)
0.6	9.40(−13)	1.94(−15)	1.92(−09)
0.7	9.93(−13)	1.97(−15)	1.54(−09)
0.8	1.03(−12)	1.86(−15)	1.53(−09)
0.9	1.04(−12)	1.89(−15)	1.85(−09)
1.0	1.05(−12)	1.89(−15)	1.85(−09)

Example 6

$$y' = y - x^2 + 1, y(0) = \tfrac{1}{2}, h = 0.1$$ (Exact solution: $y(x) = (x+1)^2 - \tfrac{1}{2}e^x$).

Source: Badmus *et al.,* (2015). Results are shown in Table 10.

Table 10: Comparison of Absolute Error for Example 6

x	HBM3	HBM4	BYP
0.1	2.1161(−13)	2.2204(−16)	−6.5410(−12)
0.2	4.6818(−13)	9.9920(−16)	−7.1360(−12)
0.3	7.7582(−13)	1.3322(−15)	−7.9700(−12)
0.4	1.1438(−12)	2.4425(−15)	−2.4400(−12)
0.5	1.5796(−12)	3.1086(−15)	−2.5670(−11)
0.6	2.0945(−12)	3.5527(−15)	7.3760(−11)
0.7	2.7016(−12)	5.5511(−15)	−3.4243(−10)
0.8	3.4115(−12)	6.2172(−15)	1.4109(−09)
0.9	4.2419(−12)	8.4377(−15)	−5.9603(−09)
1.0	5.2087(−12)	1.021(−14)	2.5051(−08)

4.4 DISCUSSION OF RESULTS AND CONCLUSION

The results of the numerical examples for HBM3 and HBM4 in this research were compared with that of Badmus & Mshelia (2.12), Yakusak & Adeniyi (2015), Yakusak, *et al.* (2015), Yahaya (2004), Odekunle *et al.*, (2012), Abdullahi, Chollom & Yahaya (2014), Badmus, Yahaya and Pam (2015), and Kamoh *et al*., (2017).
Table 5-10 give better accuracy with smaller error constants for the new HBM3 and HBM4 compared with virtually all existing methods already stated in the literature. This indicates that although these new hybrid methods are A-unstable, they have been shown to be convergent numerically as they cope effectively with non-stiff problems. Hence the methods derived are more accurate, efficient and computationally reliable.

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REFERENCES

Aboiyar, T., Luga, T. & Iyorter, B. V. (2015). Derivation of Continuous Linear Multistep Methods Using Hermite Polynomials as Basis Functions. *American Journal of Applied Mathematics and Statistics, 3(6),* 220-225. https://doi.org/10.12691/ajams-3-6-2

Abdullahi, A., Chollom, J. P & Yahaya, Y. A. (2014). A family of Two-Step block generalized Adams Methods for the solution of Non-Stiff IVP in ODEs. *Journal of Mathematical Association of Nigeria (ABACUS), 41(2A):* 67-77.

Adee, S. O. (2002). Two Approaches to Getting Started with Numerov's Method for Second Order ODEs. Master's Dissertation (Unpublished). University of Jos, Jos, Nigeria.

Akinfenwa, O. A., Jator, S. N. & Yao, N. M. (2013). Continuous block backward differentiation formula for solving stiff ordinary differential equations. *Computers and Mathematics with Applications. 65*: 996-1005. https://doi.org/10.1016/j.camwa.2012.03.111

Badmus, A. M. & Mshelia, D. W. (2012). Uniform order zero-stable Block Methods for Initial Value Problems of ordinary differential equations. *Journal of the Nigerian Association of Mathematical Physics*, *20*, 65-74.

Badmus, A. M., Yahaya, Y. A., & Pam, Y. C. (2015). Adams Type Hybrid Block Methods Associated with Chebyshev Polynomial for the Solution of Ordinary Differential Equations. *Journal of Advances in Mathematics and Computer Science*, *6*(6), 464-474. https://doi.org/10.9734/BJMCS/2015/14945

Fatunla, S. O. (1991). Block Methods for Second-order IVPs. *International Journal of Computer Mathematics. 41:* 55 – 63. https://doi.org/10.1080/00207169108804026

Henrici, P. (1962). *Discrete Variable Methods for ODEs*. John Wiley, New York, USA.

Jator, S. N. & King, K. L. (2018). Integrating oscillatory general second-order initial value problems using a block hybrid method of order 11. *Mathematical Problems in Engineering.* Article ID 3750274, 15 pages. https://doi.org/10.1155/2018/3750274

Kamoh N. M., Gyemang D. G., & Soomiyol M. C. (2017). On one justification on the use of hybrids for the solution of first order initial value problems of ordinary differential equations. *Pure and Applied Mathematics Journal*, ***6****(5):* 137-143. https://doi.org/10.11648/j.pamj.20170605.11

Kumleng, G. M., Adee, S. O., & Skwame, Y. (2013). Implicit Two Step Adam Moulton Hybrid Block Method with Two Off-Step Points for Solving Stiff Ordinary Differential Equations. *Journal of Natural Sciences Research*, ***3****(9):*77-81.

https://www.iiste.org/Journals/index.php/JNSR/article/view/7418

Lambert, J. D. (1973). *Computational Methods in Ordinary Differential Equations*. New York: John Wiley.

Odekunle, M. R., Adesanya, A. O. & Sunday, J. (2012). A New Block Integrator for the Solution of Initial Value Problems of First Order ODEs. *International Journal of Pure and Applied Sciences and Technology*, *11*(*1*): 92-100.

Yahaya, Y. A. (2004). Some Theory and Application of Continuous Linear Multi-Step Methods for Ordinary Differential Equation. Ph. D Thesis (Unpublished). University of Jos, Nigeria.

Yakusak, N. S. & Adeniyi, R. B. (2015). A Four-Step Hybrid Block Method for First Order Initial Value Problems in Ordinary Differential Equations. *AMSE JOURNALS – 2015-Series: Advances A; 52(1):* 17-30.

Yakusak, N. S., Emmanuel, S. & Ogunniran, M. O. (2015). Uniform Order Legendre Approach for continuous Hybrid Block Methods for the solution of first-order ordinary differential equations. *IOSR Journal of Mathematics (IOSR-JM), 11:* 09-14.